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While the fact that the operator in 3.2.ii is continuous follows from general Banach Algebra theory, it is interesting to see what can be proven just with complex analysis. This question leads to interesting questions in one variable complex analysis.

For $E : A(\overline{\mathbb{D}}) \to \mathbb{C}$ multiplicative and linear (we assume that $E(1) \neq 0$, which by $E(1) = E(1 \cdot 1) = E(1)E(1)$ implies $E(1) = 1$), we want to show that there is a $z_0 \in \overline{\mathbb{D}}$ such that $E(f) = f(z_0)$. Letting $z_0 = E(z)$, we immediately have that for p a polynomial, $E(p) = p(z_0)$ by linearity and multiplictivity. Furthermore, if $z_0 \notin \overline{\mathbb{D}}$, then

$$\frac{1}{z - z_0} \in A(\overline{\mathbb{D}})$$

This implies that

$$1 = E\left(\frac{z - z_0}{z - z_0}\right) = E\left(\frac{1}{z - z_0}\right)(z_0 - z_0) = 0$$

a contradiction.

For now we assume that $|z_0| < 1$. Then, for any $f \in A(\overline{\mathbb{D}})$ it is easy to see

$$\frac{f(z) - f(z_0)}{z - z_0} \in A(\overline{\mathbb{D}})$$

This implies that

$$E(f) - f(z_0) = E\left(\frac{f(z) - f(z_0)}{z - z_0}(z - z_0)\right) = E\left(\frac{f(z) - f(z_0)}{z - z_0}\right)(z_0 - z_0) = 0$$

Thus, the only remaining case is $|z_0| = 1$. I have not been able to prove this case with complex analysis methods, but it leads to the interesting question, for example, if $f \in A(\overline{\mathbb{D}})$, then (we will take $z_0 = 1$ for simplicity) is

$$\frac{f(z) - f(1)}{z - 1}$$

continuous at 1? Namely, how slowly can an analytic function continuously approach its boundary value? The above question can unfortunately be answered in the negative. For example take $f(z)$ to be the obvious branch of $\sqrt{z - 1}$, and restrict $z \in \mathbb{R}$. Thus, the limit becomes

$$\lim_{x \to 1} \frac{\sqrt{x - 1}}{x - 1}$$

which clearly is not finite.

I have been unable to overcome this difficulty, but it does lead to difficult questions. For example, for any $f \in A(\overline{\mathbb{D}})$ can one find an n, such that

$$\frac{z^n f(z) - z_0^n f(z_0)}{z - z_0} \in A(\overline{\mathbb{D}})$$

If this were true, then by the method used above we would have shown that $E = E_{z_0}$, but I am unsure of its truthfulness.